Coefficient Problem for Certain Subclass of Analytic Functions Using Quasi-Subordination

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Abstract - An analytic function $f$ is quasi-subordinate to an analytic function $g$, in the open unit disk if there exist analytic function $\varphi$ and $w$, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Certain subclasses of analytic univalent functions associated with quasi-subordination are defined and the bounds for the Fekete-Szegö coefficient functional $|a_3 - \mu a_2^2|$ for functions belonging to these subclasses are derived.

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Strictly as per the compliance and regulations of :
Coefficient Problem for Certain Subclass of Analytic Functions Using Quasi-Subordination

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Abstract - An analytic function \( f \) is quasi-subordinate to an analytic function \( g \), in the open unit disk if there exist analytic function \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = \varphi(z)g(w(z)) \). Certain subclasses of analytic univalent functions associated with quasi-subordination are defined and the bounds for the Fekete-Szego coefficient functional \( |a_3 - \mu a_2^2| \) for functions belonging to these subclasses are derived.

I. Introduction and Motivation

Let \( A \) be the class of analytic function \( f \) in the open unit disk \( D = \{z : |z| < 1\} \) normalized by \( f(0) = 0 \) and \( f'(0) = 1 \) of the form \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). For two analytic functions \( f \) and \( g \), the function \( f \) is subordinate to \( g \), written as follows:

\[
f(z) \prec g(z),
\]

if there exists an analytic function \( w \), with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \). In particular, if the function \( g \) is univalent in \( D \), then \( f(z) \prec g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(D) \subset g(D) \). For brief survey on the concept of subordination, see [1].

Ma and Minda [2] introduced the following class

\[
S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\},
\]

where \( \phi \) is an analytic function with positive real part in \( D \), \( \phi(D) \) is symmetric with respect to the real axis and starlike with respect to \( \phi(0) = 1 \) and \( \phi'(0) > 0 \). A function \( f \in S^*(\phi) \) is called Ma-Minda starlike (with respect to \( \phi \)). The class \( C(\phi) \) is the class of functions \( f \in A \) for which...
1 + zf''(z)/f'(z) \prec \phi(z). The class \( S^*(\phi) \) and \( C(\phi) \) include several well-known subclasses of starlike functions as special case.

In the year 1970, Robertson [3] introduced the concept of quasi-subordination. For two analytic functions \( f \) and \( g \), the function \( f \) is quasi-subordinate to \( g \), written as follows:

\[
f(z) \prec_q g(z),
\]

if there exist analytic function \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = \varphi(z)g(w(z)) \). Observe that when \( \varphi(z) = 1 \), then \( f(z) = g(w(z)) \), so that \( f(z) \prec g(z) \) in \( D \). Also notice that if \( w(z) = z \), then \( f(z) = \varphi(z)g(z) \) and it is said that \( f \) is majorized by \( g \) and written \( f(z) \ll g(z) \) in \( D \). Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. See [4,5,6] for works related to quasi-subordination.

Throughout this paper it is assumed that \( \varphi \) is analytic in \( D \) with \( \varphi(0) = 1 \).

Motivated by [3,4], we define the following classes.

**Definition 1.1.** Let the class \( R^*_q(\alpha, \phi) \) consists of functions \( f \in A \) satisfying the quasi-subordination

\[
\frac{z^{1-\alpha}f'(z)}{[f(z)]^{1-\alpha}} - 1 \prec_q \phi(z) - 1, \quad \alpha \geq 0
\]

**Example 1.2.** The function \( f : D \to C \) defined by the following

\[
\frac{z^{1-\alpha}f'(z)}{[f(z)]^{1-\alpha}} - 1 = z[\phi(z) - 1], \quad \alpha \geq 0
\]

belongs to the class \( R^*_q(\alpha, \phi) \).

It is well known (see [10]) that the \( n^{th} \) coefficient of a univalent function \( f \in A \) is bounded by \( n \). The bounds for coefficient give information about various geometric properties of the function. Many authors have also investigated the bounds for the Fekete-Szego coefficient for various classes [11,12,13,14,15,16,17,18,19,20,21,22,23,24,25]. In this paper, we obtain coefficient estimates for the functions in the above defined classes.

Let \( \Omega \) be the class of analytic functions \( w \), normalized by \( w(0) = 0 \), and satisfying the condition \( |w(z)| < 1 \). We need the following lemma to prove our results.

**Lemma 1.3** (see [26]). If \( w \in \Omega \), then for any complex number \( f \)

\[
|w_2 - tw_2^2| \leq \max\{1; |t|\}.
\]

The result is sharp for the functions \( w(z) = z^2 \) or \( w(z) = z \).
Throughout, let \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \) \( \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \) \( \varphi(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \) \( B_1 \in \mathbb{R} \) and \( B_1 > 0. \)

**Theorem 2.1.** If \( f \in A \) belongs to \( R_\ell^\ast(\alpha, \phi) \), then

\[
|a_2| \leq \frac{B_1}{1 + \alpha},
\]

\[
|a_3| \leq \frac{B_1}{2} \left( 1 + \max \left\{ 1, B_1 \left| \frac{1 - \alpha}{1 + \alpha} + \frac{\alpha}{2B_1} \right| \right\} \right) (2.1)
\]

and for any complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{2} \left( 1 + \max \left\{ 1, B_1 \left| \frac{1 - \alpha}{1 + \alpha} - \frac{2\mu}{(1 + \alpha)^2} + \frac{\alpha}{2B_1} \right| \right\} \right). (2.2)
\]

**Proof.** If \( f \in R_\ell^\ast(\alpha, \phi) \), then there exist analytic functions \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that

\[
\frac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1 = \varphi(z)(\phi(w(z)) - 1). \quad (2.3)
\]

Since

\[
\varphi(w(z)) - 1 = B_1 c_0 w_1 z + (B_1 c_1 w_1 + c_0 (B_1 w_2 + B_2 w_1^2)) z^2 + \cdots \quad (2.4)
\]

it follows from (2,3) that

\[
a_2 = \frac{B_1 c_0 w_1}{(1 + \alpha)}
\]

\[
a_3 = \frac{1}{2 + \alpha} \left[ \frac{\alpha}{2} B_1 c_0 w_1 + B_1 c_1 w_1 + B_1 c_0 w_2 + c_0 \left( \left( \frac{1 - \alpha}{1 + \alpha} \right) B_1^2 c_0 + B_2 \right) w_1^2 \right] \quad (2.5)
\]

Since \( \varphi(z) \) is analytic and bounded in \( D \), we have [27, page 172]

\[
|c_n| \leq 1 - |c_0|^2 \leq 1 \quad (n > 0). \quad (2.6)
\]

By using this fact and the well-known inequality, \( |w_1| \leq 1 \), we get

\[
|a_2| \leq \frac{B_1}{1 + \alpha}. \quad (2.7)
\]
Further,
\[ a_3 - \mu a_2^2 = \frac{1}{2 + \alpha} \left[ B_1 c_1 w_1 + c_0 (B_1 w_2 + \frac{\alpha}{2} B_1 w_1) \\ + \left( B_2 + \left( 1 - \frac{\alpha}{1 + \alpha} \right) B_1^2 c_0 - \frac{2\mu}{(1 + \alpha)^2} B_1^2 c_0 \right) w_1^2 \right] . \] (2.8)

Then
\[ |a_3 - \mu a_2^2| \leq \frac{1}{2 + \alpha} \left( |B_1 c_1 w_1| \right. \\
+ \left. |B_1 c_0 \left( w_2 - \left( \frac{2\mu}{(1 + \alpha)^2} B_1 c_0 - \left( 1 - \frac{\alpha}{1 + \alpha} \right) B_1 c_0 + \frac{\alpha w_1}{2} c_0 - \frac{B_2}{B_1} \right) w_1^2 \right) | \right) . \] (2.9)

Again applying \(|c_n| \leq 1|\) and \(|w_1| \leq 1|\), we have
\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{2 + \alpha} \left( 1 + |w_2 - \left( \frac{\alpha}{2} - \left( 1 - \frac{\alpha}{1 + \alpha} - \frac{2\mu}{(1 + \alpha)^2} \right) B_1 c_0 - \frac{B_2}{B_1} \right) w_1^2 | \right) . \] (2.10)

Applying Lemma 1.3 to
\[ |w_2 - \left( \frac{\alpha}{2} - \left( 1 - \frac{\alpha}{1 + \alpha} - \frac{2\mu}{(1 + \alpha)^2} \right) B_1 c_0 - \frac{B_2}{B_1} \right) w_1^2 | \] yields
\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{2 + \alpha} \left( 1 + \max \left\{ 1, \left( \frac{\alpha}{2} - \left( 1 - \frac{\alpha}{1 + \alpha} - \frac{2\mu}{(1 + \alpha)^2} \right) B_1 c_0 - \frac{B_2}{B_1} \right) \right\} \right) . \] (2.11)

Observe that
\[ \left| \frac{\alpha}{2} - \left( 1 - \frac{\alpha}{1 + \alpha} - \frac{2\mu}{(1 + \alpha)^2} \right) B_1 c_0 - \frac{B_2}{B_1} \right| \leq B_1 \left| c_0 \right| \left| 1 - \frac{\alpha}{1 + \alpha} - \frac{2\mu}{(1 + \alpha)^2} + \frac{\alpha}{2B_1} \right| + \left| \frac{B_2}{B_1} \right| , \] (2.12)

and hence we can conclude that
\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{2} \left( 1 + \max \left\{ 1, B_1 \left| 1 - \frac{\alpha}{1 + \alpha} - \frac{2\mu}{(1 + \alpha)^2} + \frac{\alpha}{2B_1} \right| + \left| \frac{B_2}{B_1} \right| \right\} \right) . \] (2.13)

For \( \mu = 0 \), the above will reduce to estimate of \(|a_3|\).

**Theorem 2.2.** If \( f \in A \) satisfies
\[ \frac{z^{1-\alpha} f'(z)}{|f(z)|^{1-\alpha}} - 1 \ll \phi(z) - 1, \] (2.15)
then the following inequalities hold:

\[ |a_2| \leq \frac{B_1}{1 + \alpha}, \]

\[ |a_3| \leq \frac{1}{2 + \alpha}(B_1 + B_1^2 + |B_2|), \tag{2.16} \]

and, for any complex number \( \mu \),

\[ |a_3 - \mu a_2^2| \leq \frac{1}{(2 + \alpha)(1 + \alpha)^2}((1 + \alpha)^2B_1 + |(1 + \alpha)^2 - (2 + \alpha)\mu|B_1^2 + (1 + \alpha)^2|B_2|). \tag{2.17} \]

**Proof.** The result follows by taking \( w(z) = z \) in the proof of Theorem 2.1.

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### References Références Referencias


